Article

Directed transport and scaling laws in a nontwist Hamiltonian mapping

Matheus Rolim Sales[1]

Daniel Borin[1], Leonardo Costa de Souza[2], José Danilo Szezech Jr.[3], Ricardo Luiz Viana[4], Iberê Luiz Caldas[2], Edson Denis Leonel[1]

DDAYS LAC 2024

Motivation

The phase space of a two-dimensional integrable Hamiltonian system features periodic and quasiperiodic invariant tori. According to the Kolmogorov-Arnold-Moser (KAM) theorem, weak perturbations preserve irrational tori (KAM tori) but destroy rational ones, leading to chaotic motion near hyperbolic fixed points and stability islands around elliptical points. Increasing perturbations further break down KAM tori into cantori, which act as partial barriers and cause intermittent quasiperiodic behavior known as stickiness. This complex structure impacts the transport and statistical properties of particles in phase space. Recently, the Hamiltonian ratchet effect, a preferential transport direction without external bias, has been observed. In this study, we examine transport and diffusion in a nontwist mapping relevant to toroidal plasmas. We analyze survival probabilities, escape dynamics, and recurrence times, demonstrating unbalanced stickiness in phase space and confirming the ratchet effect. Additionally, we explore diffusion scaling properties, providing a robust analysis of observed scaling invariance.

The model

We consider the following nontwist mapping introduced in the context of **E**×**B** drift in toroidal plasmas:

$$I_{n+1} = I_n + \varepsilon \sin(2\pi\theta_n),$$
$$\theta_{n+1} = \theta_n + \mu v(I_{n+1})\left[\frac{M}{q(I_{n+1})} - L\right] + \rho\frac{E(I_{n+1})}{\sqrt{|I_{n+1}|}},$$

where $q(I)$, $E(I)$, and $v(I)$ are given by, respectively,

$$q(I) = q_1 + q_2 I^2 + q_3 I^3,$$
$$E(I) = e_1 I + e_2\sqrt{|I|} + e_3,$$
$$v(I) = v_1 + v_2 \tanh(v_3 I + v_4).$$

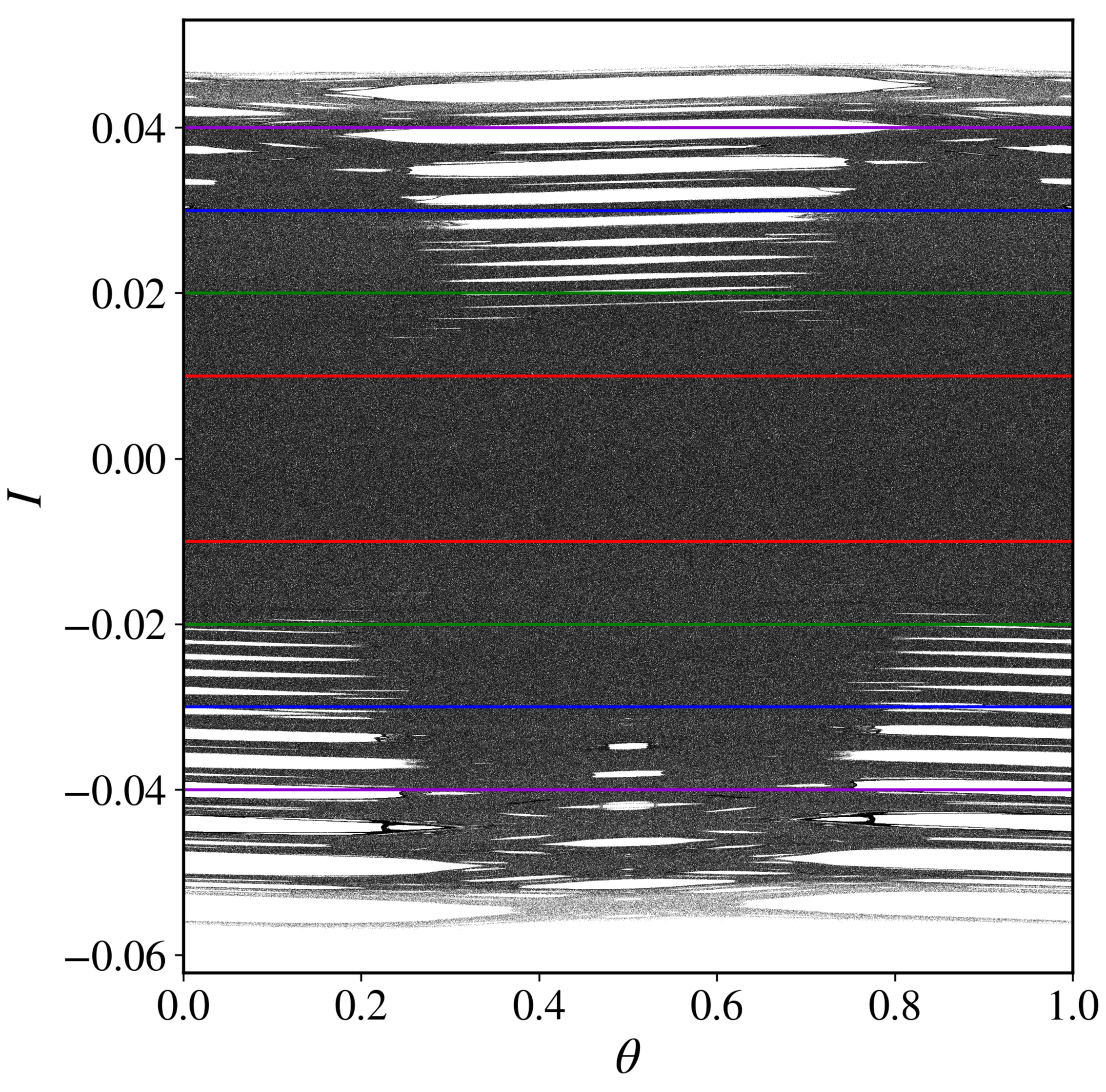

Figure 1. The phase space for $\varepsilon = 1.0 \times 10^{-3}$.

Directed transport

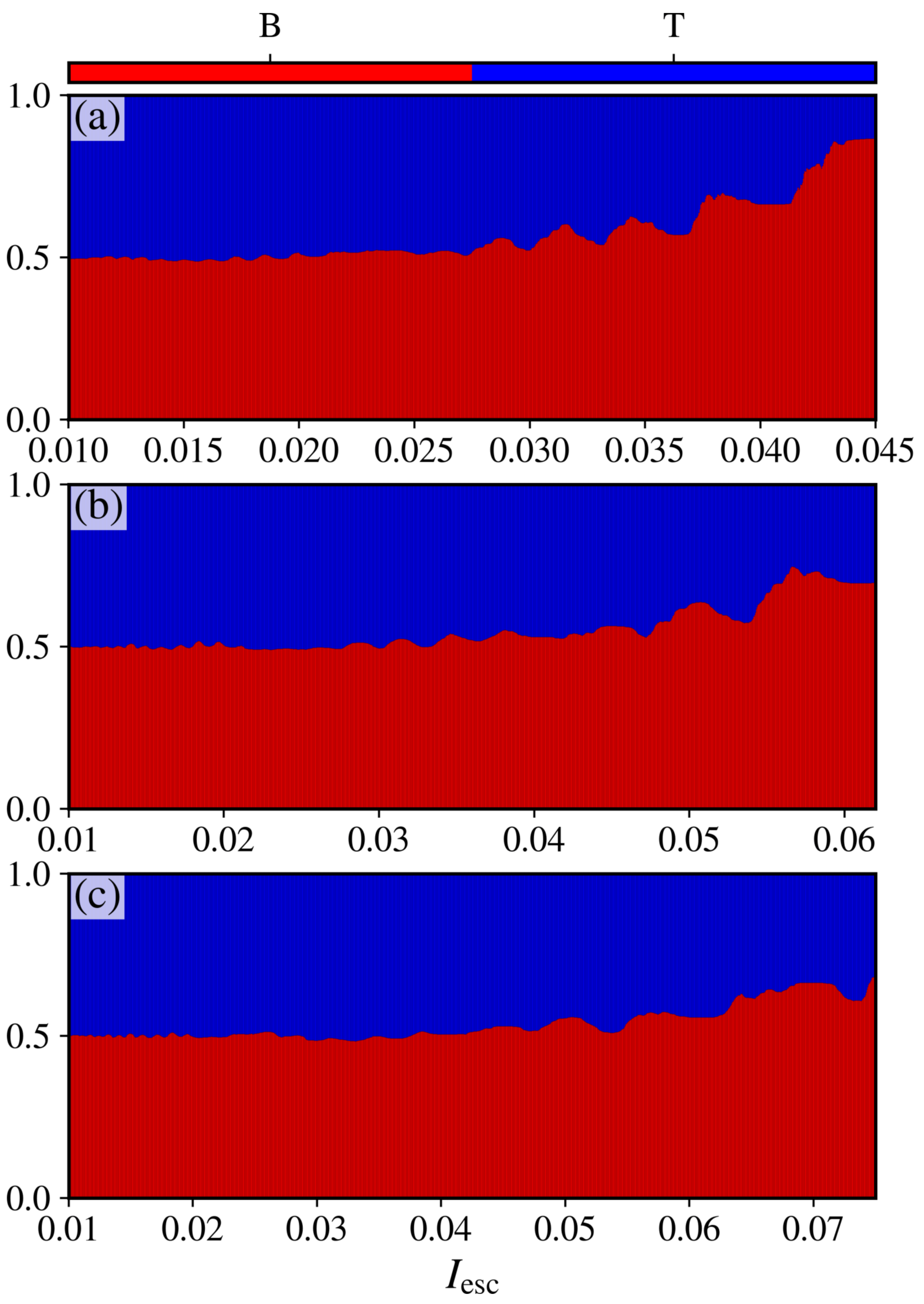

Figure 2. Fraction of initial conditions that escape through the (red) bottom exit and the (blue) top exit as a function of the exits position with (a) $\varepsilon = 1.0\times10^{-3}$, (b) $\varepsilon = 2.0\times10^{-3}$, and (c) $\varepsilon = 3.0\times10^{-3}$.

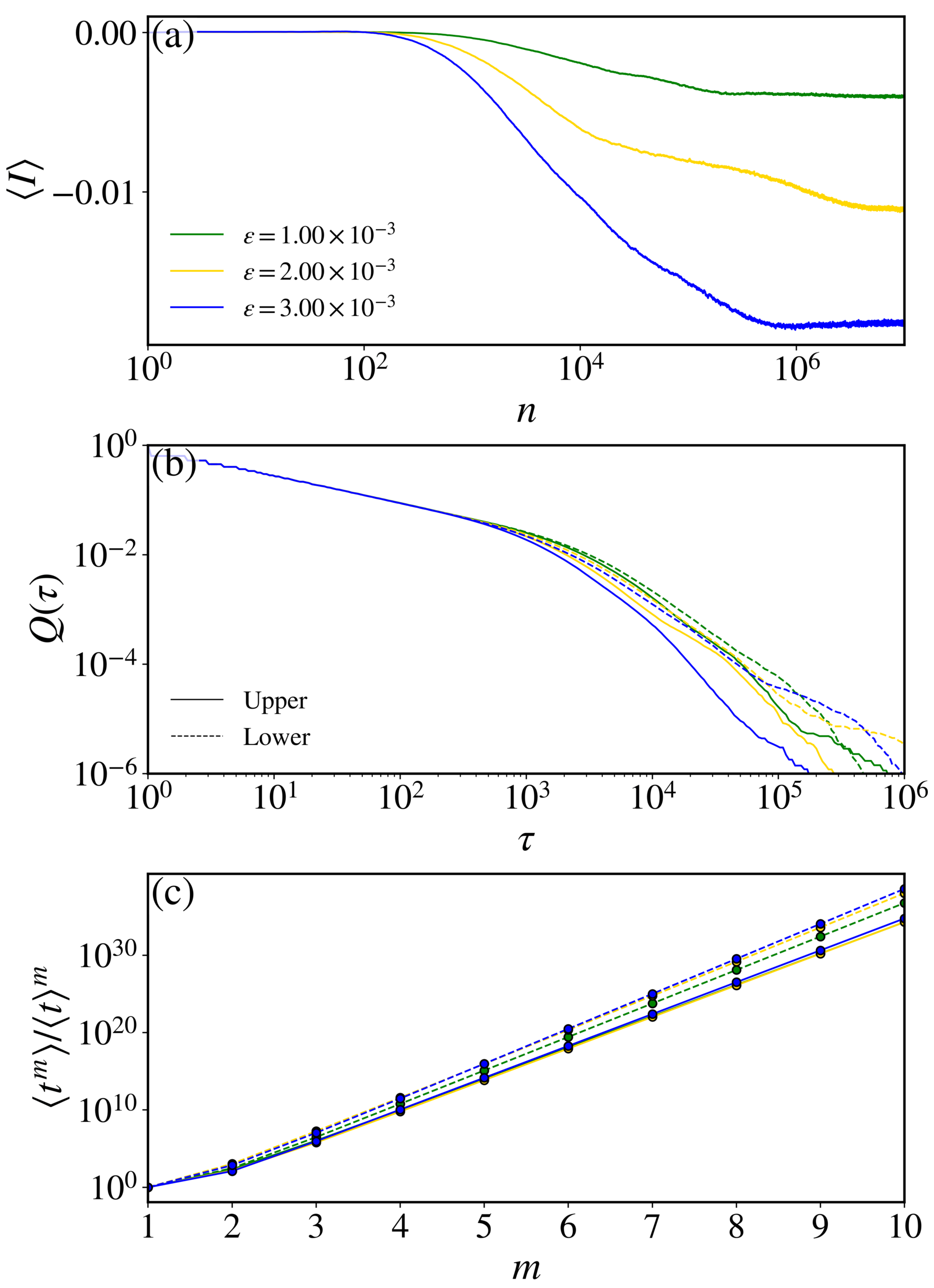

Figure 3. (a) The average of the action for an ensemble of 10^6 initial conditions randomly distributed on the line $I = 1.0\times10^{-10}$ at $n = 0$ for different values of ε. (b) The cumulative distribution of recurrence times for the (full lines) top ($I > 0$) and (dashed lines) bottom ($I < 0$) regions of phase space. (c) Moments of the distribution of recurrence times normalized to $\langle t \rangle$.

Scaling law

The observable of interest is the square root of the averaged squared action, defined by

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\frac{1}{n}\sum_{j=1}^{n} I_{i,j}^2}.$$

The behavior of I_{rms} can be characterized by four critical exponents, α, β, γ, and z assuming the following scaling hypotheses:

1. $I_{\mathrm{rms}} \sim (n\varepsilon^{\alpha})^{\beta}$ for $n \ll n_x$,
2. $I_{\mathrm{rms,sat}} \sim \varepsilon^{\gamma}$ for $n \gg n_x$,
3. $n_x \sim \varepsilon^{z}$.

These critical exponents are related through the following scaling law

$$z = \frac{\gamma}{\beta} - \alpha.$$

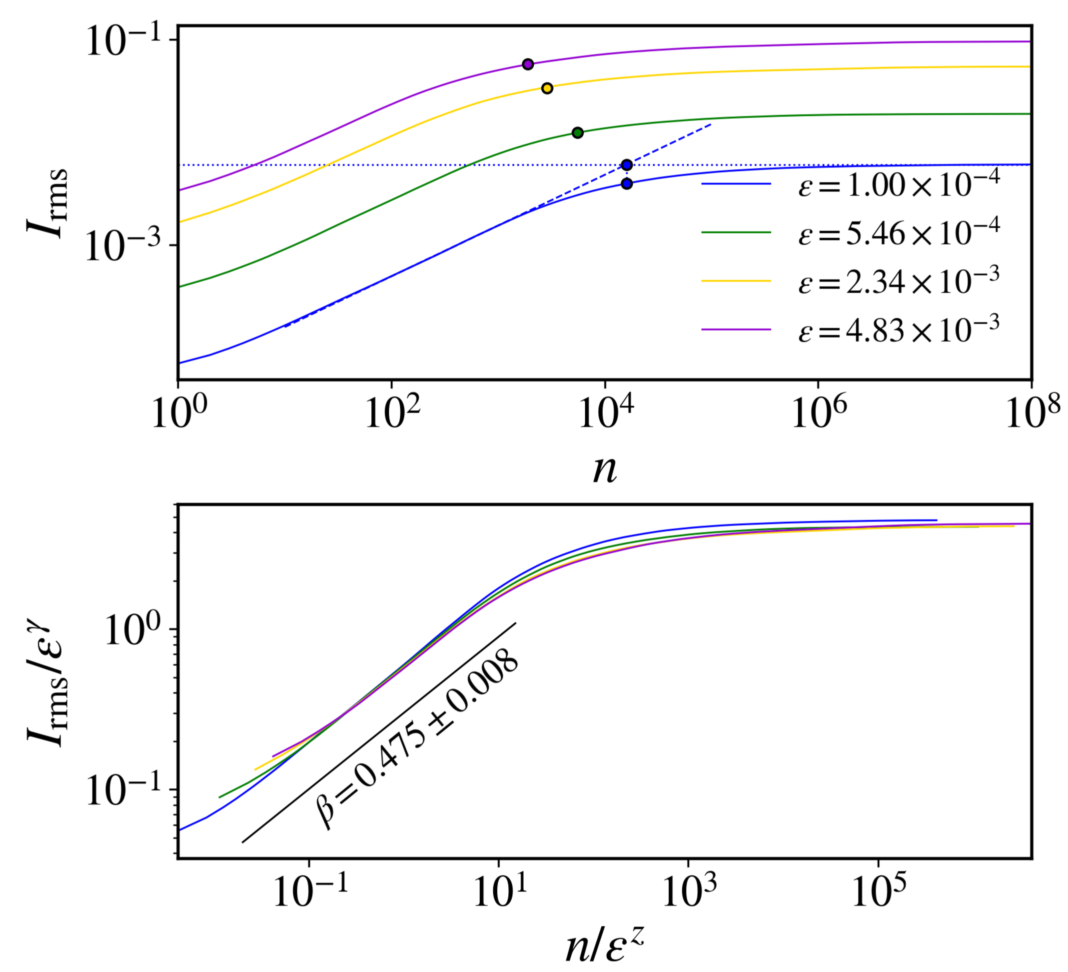

Figure 4. (a) The behavior of I_{rms} for different values of ε. The colored dots indicate the transition point from the growth regime to the constant plateau of staturation. (b) The overlap of all curves into a single, and hence, universal curve after the transformation $n \to n/\varepsilon^{z}$ and $I_{\mathrm{rms}} \to I_{\mathrm{rms}}/\varepsilon^{\gamma}$.

Summary and overview

- We have demonstrated that particles escape evenly through both the bottom and top exits when there are no stability islands in the escape region.
- The presence of stability islands in the escape region creates an unbalanced stickiness effect, which leads to the generation of a ratchet current in phase space.
- This unbalanced stickiness is caused by the inherent asymmetry of the system, resulting in different distributions of recurrence times in the top ($I > 0$) and bottom ($I < 0$) regions of phase space.
- Based on three scaling hypotheses, we have found that diffusion in phase space is characterized by four critical exponents.
- We have derived a scaling law that relates these four exponents, and through extensive numerical simulations, we have obtained all of them, showing remarkable agreement with the scaling law.

AFFILIATIONS

[1]Department of Physics, São Paulo State University (UNESP), 13506-900, Rio Claro, SP, Brazil
[2]Institute of Physics, University of São Paulo, 05508-900, São Paulo, SP, Brazil
[3]Department of Mathematics and Statistics, State University of Ponta Grossa, 84030-900, Ponta Grossa, PR, Brazil
[4]Department of Physics, Federal University of Paraná, 80060-000, Curitiba, PR, Brazil

ACKNOWLEDGEMENTS

FAPESP — Fundação de Amparo à Pesquisa do Estado de São Paulo
unesp
CNPq
Grupo de Investigação — Sistemas Complexos e Dinâmica Não Linear
105 GROUP

MORE INFORMATION

Matheus Rolim Sales
Postdoctoral researcher, São Paulo State University (UNESP), Department of Physics, Rio Claro, SP, Brazil
rolim.sales@unesp.br
0000-0002-1121-6371

https://mrolims.github.io/