Characterizing stickiness using recurrence time entropy

Article

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Motivation

In two-dimensional quasi-integrable Hamiltonian systems with hierarchical phase space, chaotic orbits can spend an arbitrarily long time around islands, in which they behave similarly as quasiperiodic orbits. This phenomenon is called stickiness, and it is due to the presence of partial barriers to the transport around the hierarchical levels of islands-around-islands. The stickiness affects the convergence of the Lyapunov exponents, making the task of characterizing the dynamics more difficult, especially when only short time series are known. Due to the intrinsic property of dynamical systems that quasiperiodic orbits lying on invariant circles can have at most three different return times, we propose the use of the recurrence time entropy (RTE) (estimated from the recurrence plots) to characterize the dynamics of nonlinear systems.

The standard map

The standard map is an area-preserving map, and its dynamics is given by

$$\begin{aligned} p_{n+1} &= p_n - k\sin\theta_n, \\ \theta_{n+1} &= \theta_n + p_{n+1}, \end{aligned} \tag{1}$$

where p_n and θ_n are the canonical momentum and position, respectively, at discrete times, and k controls the nonlinearity of the system. In spite of its simple mathematical form, the standard map exhibits all the features of a typical quasi-integrable Hamiltonian system.

Recurrence plots

The recurrence plot (RP) is a graphical representation of the recurrences of a time series of a given dynamical system. Given a trajectory $\mathbf{x}_i \in \mathbb{R}^d$, with $i = 1, 2, \ldots, N$ and N is the length of the times series, the recurrence matrix is defined as

$$R_{ij} = H(\epsilon - \|\mathbf{x}_i - \mathbf{x}_j\|), \tag{2}$$

where $H(\cdot)$ is the Heaviside unit step funtion, ϵ is a small threshold and $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the states i and j in the d-dimensional phase space in terms of a suitable norm.

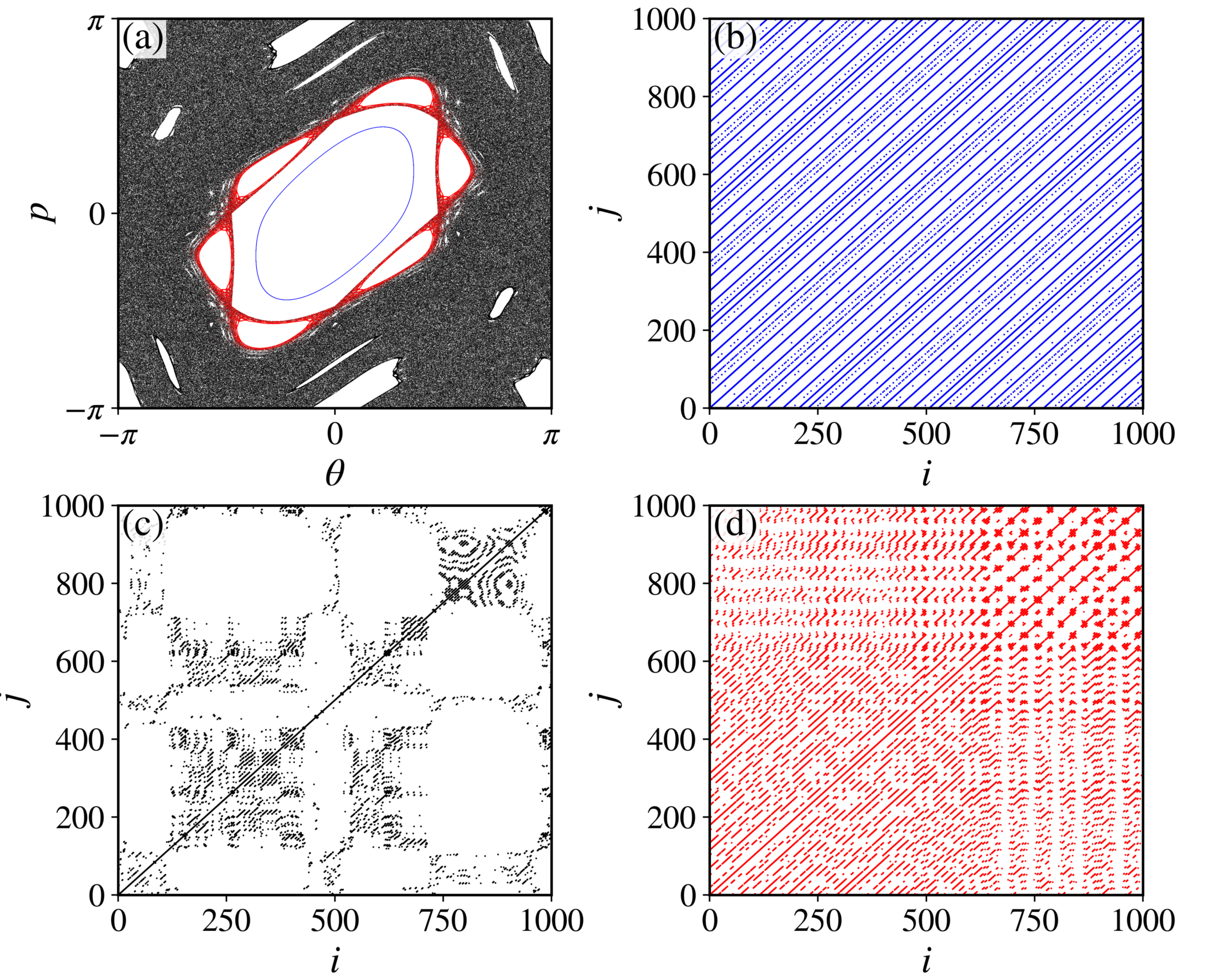

Figure 1. (a) (blue) A quasiperiodic orbit, (black) a chaotic orbit, and (red) a sticky orbit of the standard map with $k = 1.5$. (b)-(d) are the corresponding recurrence matrices for the first 1000 iterates.

The recurrence time entropy

The white vertical lines in an RP are an estimate of the recurrence times of an orbit. Therefore, we compute the recurrence time entropy (estimated from the RP), RTE, to characterize the dynamics of the orbit as follows:

$$\mathrm{RTE} = -\sum_{\ell=\ell_{\min}}^{\ell_{\max}} p(\ell)\ln p(\ell), \tag{3}$$

where, $p(\ell) = P(\ell)/N_\ell$ is the relative distribution of white vertical lines with length ℓ. For periodic orbits, we expect RTE $= 0$. A quasiperiodic orbit has a low value of RTE, whereas a chaotic orbit is characterized by a large value of RTE.

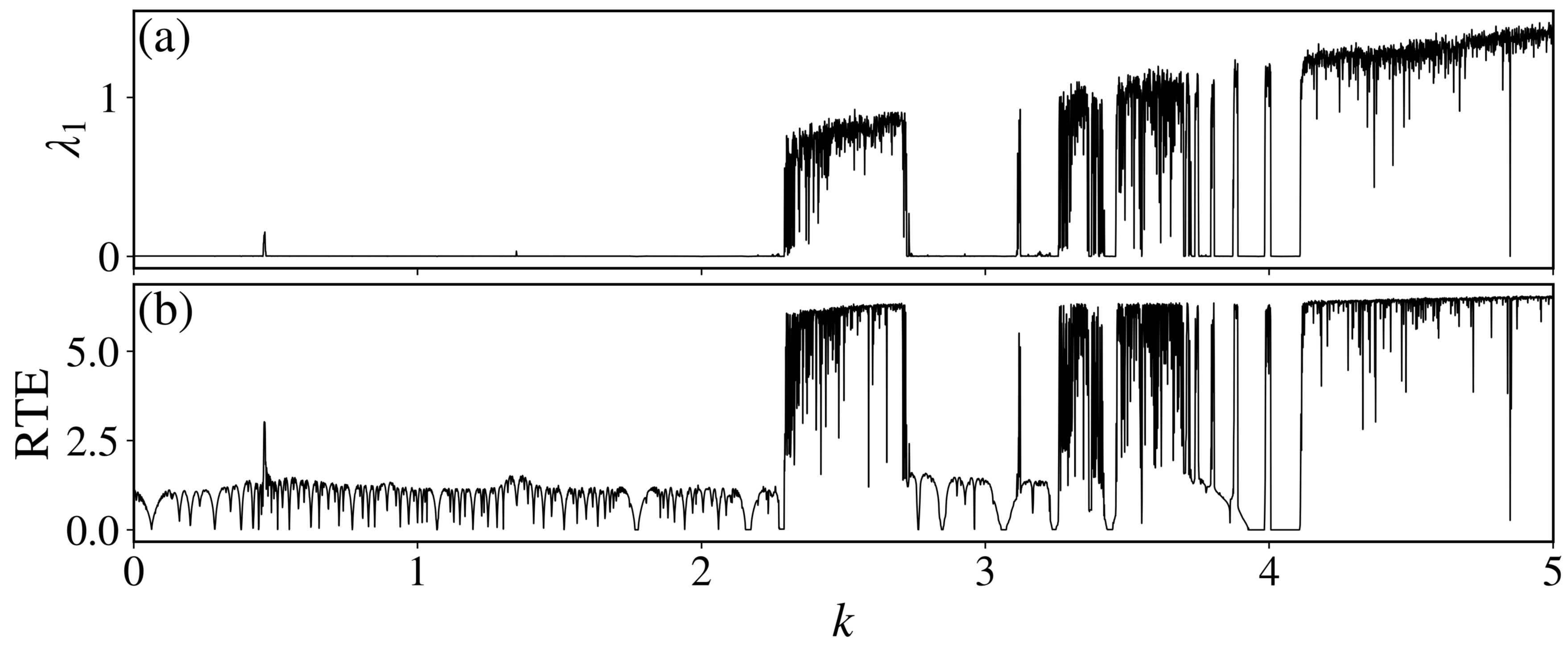

Figure 2. (a) The largest Lyapunov exponent and (b) the RTE for the standard map [Eq. (1)], as a function of the nonlinearity parameter k with initial condition $(\theta_0, p_0) = (0.0, 1.3)$ and time series length of $N = 5000$. The correlation coefficient between λ_1 and RTE is $\rho = 0.95$ for the data in (a) and (b).

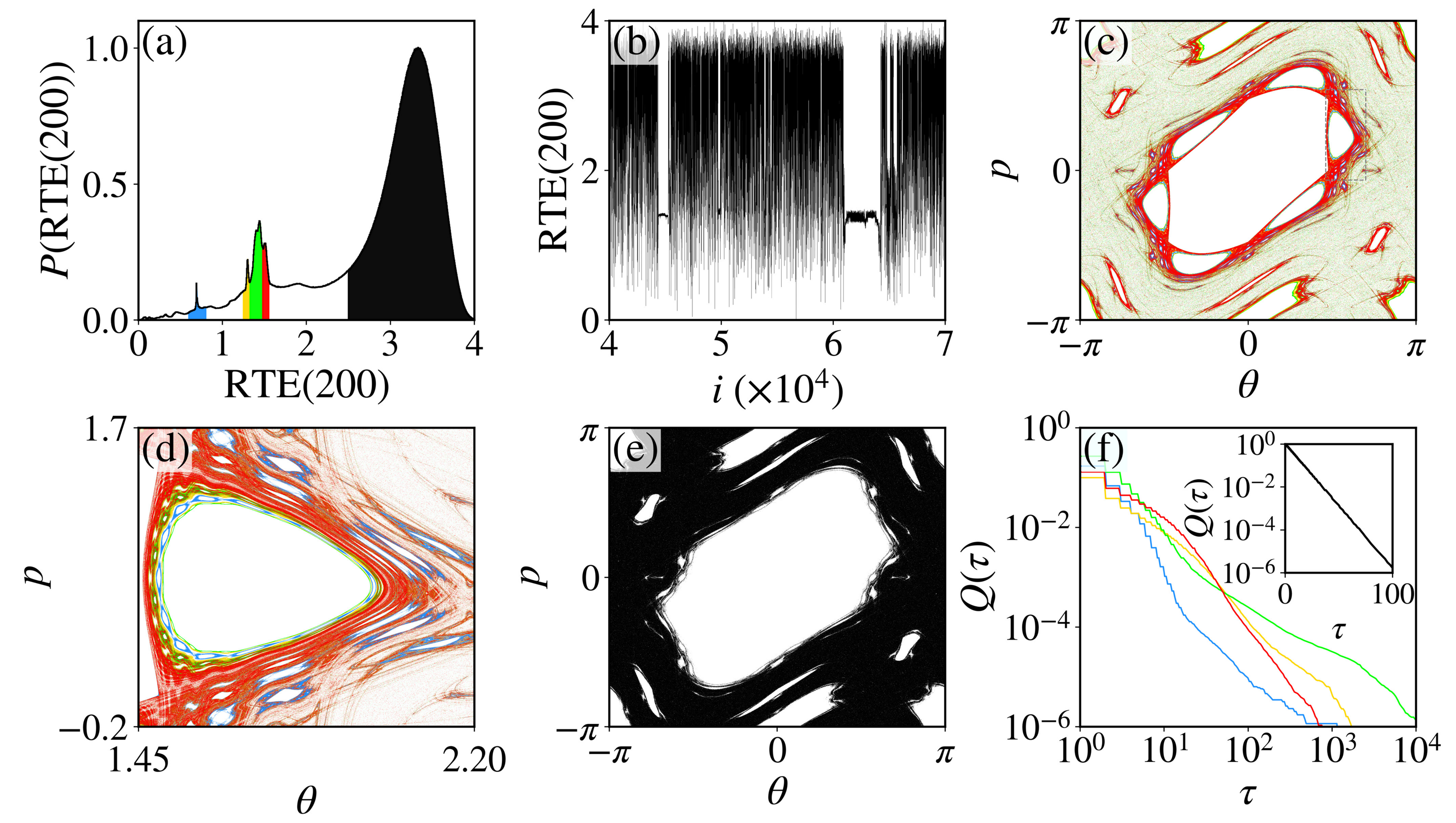

Figure 3. (a) The finite-time RTE distribution for a single chaotic orbit with $k = 1.5$. (b) The "time series" of the finite-time RTE shown in (a). (c) The phase space points that generate the colored peaks in (a); (d) is a magnification of one of the period-6 satellite islands in (c), indicated by the gray dashed lines. (e) The phase space points that generate the larger (black) peak in (a). (f) The cumulative distribution of trapping times, $Q(\tau)$, for each sticky region identified in (a). The inset is a log-lin plot of $Q(\tau)$ for the trapping times that generate the larger (black) peak in (a).

Summary and overview

- It is possible to distinguish between chaotic and regular solutions using the RTE.
- The peak for small values of λ_1 in the finite-time Lyapunov exponent distribution is, in fact, composed of several minor peaks, as suggested by Harle and Feudal.
- Each peak in the finite-time RTE distribution corresponds to a different hierarchical level in the islands-around-islands structure embedded in the chaotic sea.
- Can the RTE characterize the parameter space of dissipative systems?
- Can the RTE characterize the dynamics of higher dimensional systems?
- Can we define an upper bound for the RTE?

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MORE INFORMATION

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